Copy No. 4

~~NEW YORK UNIVERSITY~~
~~INSTITUTE OF MATHEMATICAL SCIENCES~~
~~LIBRARY~~
~~25 Waverly Place, New York 3, N.Y.~~

IMM-NYU 199
July 1953

IMM 199

NEW YORK UNIVERSITY

~~NEW YORK UNIVERSITY~~
~~INSTITUTE OF MATHEMATICAL SCIENCES~~
~~LIBRARY~~
~~4 Washington Place, New York 3, N.Y.~~

NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY
251 Mercer St. New York, N.Y. 10012

Decay of Drops by Evaporation

by

IGNACE I. KOLODNER

Institute for Mathematics and Mechanics

Prepared under Contract No. DA-18-064-CML-1860
with United States Army Chemical Corps

IMM-199
c.1

NEW YORK UNIVERSITY

Institute for Mathematics and Mechanics

DECAY OF DROPS BY EVAPORATION

by

Igance I. Kolodner

This Report Represents Results Obtained at the Institute of Mathematical Sciences, New York University, under the Auspices of the United States Army Chemical Corps, Contract No. DA-18-064-CML-1860

New York, 1953

opp

Manufactured in the United States by the Office of
Publications and Printing of New York University

DECAY OF DROPS BY EVAPORATION

Ignace I. Kolodner

1. Introduction

In order to determine the radius $R(t)$ of an evaporating drop it is necessary to simultaneously consider the concentration of vapor $c(r,t)$ around the drop. If we assume that the temperature remains constant and the drop remains spherical, then $c(r,t)$ and $R(t)$ can be determined by solving the following problem:

$$
\begin{aligned}
(1) \qquad & c_{rr} + \frac{2}{r} c_r = c_t && \text{for} \quad r > \max\,[0, R(t)]\,, \quad t > 0 \\
& c[R(t),t] = 1 && \text{for} \quad 0 \leq t \leq t_0 \\
& \alpha c_r[R(t),t] = \dot{R}(t) && \text{for} \quad 0 < t < t_0 \\
& c(r,0) = 0 && \text{for} \quad r > 1 \\
& c(\infty,t) = 0 && \text{for} \quad t \geq 0 \\
& R(0) = 1 && \\
& c_r(0,t) = 0 && \text{for} \quad t > t_0 \quad .
\end{aligned}
$$

In equations (1) all quantities are in the dimensionless form given in [1] (see equations (7)-(12) of [1]). The constant t_0 is the time required for complete evaporation, and is to be determined, while α is a given constant.

The main results of [1] are that over most of the evaporation time the evaporation curve is approximately given by

$$
(2) \qquad R^2(t) = 1 - 2\alpha t - \frac{4}{\sqrt{\pi}}\,\alpha\sqrt{t}
$$

and that the time of evaporation t_0 is approximately

(3) $$t_o = \frac{1}{2\alpha}\left[1 - \frac{8}{3\sqrt{2\pi}}\sqrt{\alpha} + O(\alpha)\right] .$$

Although these results agree well with experiment and with previous approximate results, they are based on a purely formal approximation method. Therefore, in the present report the problem is treated rigorously by a different method.

The main results of the present report are:

(a) The problem (1) possesses a unique solution if $0 \le \alpha \le \frac{1}{1+\sqrt{8/\pi}} \approx .38.$

(b) $R(t)$ is the limit of the sequence $\rho_n(t)$ constructed in section 3.

(c) For every n, $\rho_{2n}(t) \le R(t) < \rho_{2n+1}(t)$. Thus upper and lower bounds for $R(t)$ are obtained.

(d) $1 - 2\alpha(t + 2\sqrt{\frac{t}{\pi}}) \le R^2(t) \le 1 - \frac{2\alpha}{\alpha+1}t$.

(e) $\frac{1}{2\alpha} + \frac{2}{\pi}(1 - \sqrt{1+\frac{\pi}{2\alpha}}) < t_o < \frac{1}{2\alpha} + \frac{1}{2}$.

The bounds on $R^2(t)$ in (d) are the crudest of those described in (c). Nevertheless, the lower bound coincides with equation (2), the result of [1], and if the lower bound on t_o in (e) is expanded for small α, it coincides with equation (3).

In Figure 1 the two bounds in (d) are plotted for $\alpha = .05$, and an improved upper bound is also shown. A large value of α has purposely been chosen in order to show the differences between the bounds. In a practical problem in which α is of the order 10^{-5} the various bounds could not be distinguished from each other, and therefore either of them would represent the solution adequately.

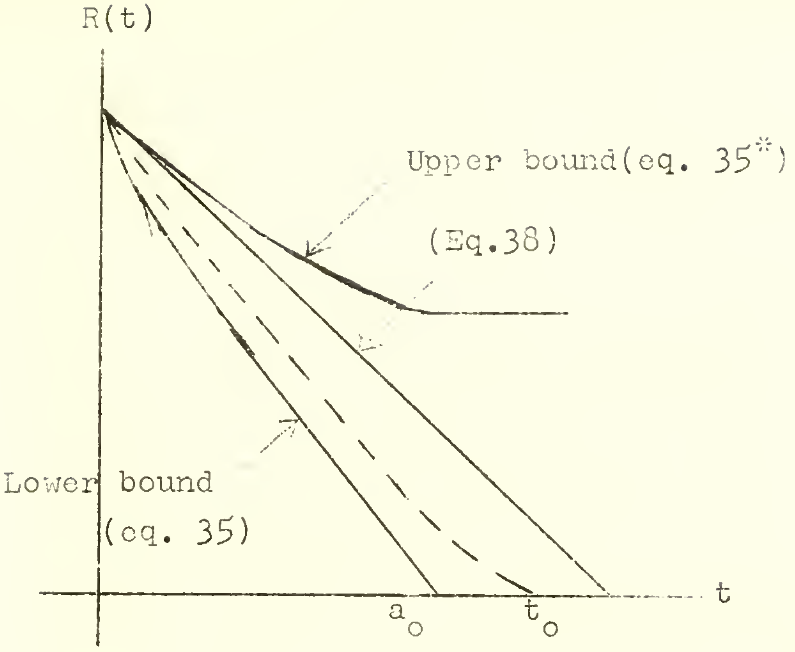

Fig. I

The difference between the two bounds in (d) is less than $\frac{\alpha}{\alpha+1}(1+4\sqrt{\frac{a_o}{\pi}})$ during the full time of evaporation, where a_o is the lower bound in (c). Similarly, the bounds in (c) determine t_o with a relative error of less than $\left[\frac{1+\alpha}{2\alpha} - a_o\right]a_o^{-1}$. For a water droplet evaporating in air at 73°F, $\alpha \sim 2 \times 10^{-5}$ and these errors are .0071 and .0072 respectively.

The restriction on α in (a) is a consequence of the method of solution. In fact, for $0 \leq \alpha < \frac{\pi}{4+\sqrt{\frac{2\pi}{e}}} \approx .57$ the method still yields a solution but not for all values of t up to t_o. The solution so obtained would have to be extended up to t_o by another method. Most likely the solution does exist for all $\alpha \geq 0$, but the present method will not apply for all α, since it can be shown to fail if $.90 < \alpha < 1.15$. In the corresponding one-dimensional problem the present method does yield a solution for all $\alpha \geq 0$.

If $\alpha < 0$, the vapor condenses and the drop grows, so $\dot{R} > 0$. Therefore, Theorem 3 of section 2 is not applicable, and the present method must be modified. A procedure for doing this has been found, and will be reported in the near future. In the one-dimensional case the exact solution is known for all $\alpha > -1$ and therefore the same restriction on α must be expected in the spherical case.

In section 2 we derive an integral representation for the condensation depending on the evaporation curve $R(t)$, and show that this expression satisfies all of the conditions imposed on $c(r,t)$, equations (1), provided that $R(t)$ is the proper evaporation curve. $R(t)$ is determined as a solution of a certain integro-differential equation. In section 3, the equation for $R(t)$ is solved by constructing a sequence of functions which converge to the solution. In section 4 we derive simple lower and upper bounds for $R(t)$. (see above) Finally, section 5 contains some comments concerning the method and its applicability to other problems of similar structure.

The proof of statements appearing in sections 2-4 is in some cases long and intricate. We therefore eliminated all those proofs from the text, in order not to obstruct its clarity. They will be published in a future issue of Communications on Pure and Applied Mathematics, New York University. In the same spirit, statements of theorems have been simplified in some cases so as to eliminate certain "minor" points.

2. Integral representation for c(r,t)

In place of $c(r,t)$ it is more convenient to consider the function

$$u(r,t) = rc(r,t) \quad . \tag{4}$$

From (1) it follows that u satisfies the following conditions:

(5) $u_{rr} = u_t$, $r > \max(0, R(t))$, $t > 0$

(6) $u(R(t),t) = R(t)$, $0 \leq t \leq t_o$

(7) $\alpha u_r(R(t),t) = R(t)\dot{R}(t) + \alpha$, $0 < t < t_o$

(8) $u(r,0) = 0$, $r > 1$, $u(\infty,t) = 0$

(9) $R(0) = 1$,

(10) $\lim\limits_{r \to 0} \frac{u}{r}$ exists and is bounded for $t > t_o$,

(11) $\lim\limits_{r \to 0} \frac{ru_r - u}{r^2} = 0$, $t > t_o$.

We first consider the problem in the range $0 \leq t \leq t_o$ and consequently disregard the conditions (10) and (11) which apply only for $t > t_o$.

Let $U(r,t,\xi,\sigma)$ be the fundamental solution of the heat equation

$$U = \frac{1}{2\sqrt{\pi(t-\sigma)}} \exp\left(-\left(\frac{r-\xi}{2\sqrt{t-\sigma}}\right)^2\right) \quad . \tag{12}$$

If the solution to our problem were known to exist and if the boundary function $R(t)$ were known, then $u(r,t)$ could be obtained from the well-known representation of solutions of the heat equation by line integrals involving boundary values, viz:

$$u(r,t) = \oint\limits_{\sigma \leq t} u(\xi,\sigma)\, U d\xi + \left(U \frac{\partial u}{\partial \xi} - u \frac{\partial U}{\partial \xi}\right) d\sigma \quad , \tag{13}$$

see e.g., [2], vol. III, p. 311, formula (35). Substituting the data (6), (7) and (8) into (13) we obtain the formal expression

$$(14)\qquad u(r,t) = \frac{1}{2}\int_0^t \left(\frac{R(\sigma)}{t-\sigma}z - \frac{(1+\alpha^{-1})R(\sigma)\dot{R}(\sigma)+1}{\sqrt{t-\sigma}}\right)\eta(z)\,d\sigma \quad ,$$

where

$$(15)\qquad z = z(r,\sigma) = \frac{r-R(\sigma)}{2\sqrt{t-\sigma}} \quad ,$$

$$(16)\qquad \eta(z) = \frac{1}{\sqrt{\pi}}\exp(-z^2) \quad ,$$

and $r > R(t)$, $t > 0$.

It is clear that expression (14) with arbitrary $R(t)$ cannot satisfy all the conditions imposed on u, since then the boundary would be prescribed, and by a well-known theorem a solution of the heat equation is uniquely determined by specifying one boundary condition only, and not two as in the present case, see equations (6) and (7). One way to proceed would be to determine $R(t)$ by requiring that either one of (6) or (7) be satisfied. This apparently simple procedure presents a number of difficulties. Firstly, it is not clear whether equation (14) or its derivatives could be used for $r = R(t)$, since its value on the boundary (if it exists at all) may not be equal to the limit from the interior. Secondly, if the first difficulty could be overcome, the resulting equation for $R(t)$ would not be easy to solve. Lastly, even if $R(t)$ could be so determined, it would again be difficult to show that the second of the two conditions (6) or (7) is then satisfied.

The object of the remainder of this section is to determine an equation for $R(t)$ and to show that with the proper $R(t)$ (14) actually represents the solution to our problem in the evaporation stage. To this effect, we consider an arbitrary smooth function $\rho(t)$ which decreases monotonically from unity,

$$(17)\qquad \rho(0) = 1 \quad , \quad \dot{\rho}(t) < 0$$

($\dot{\rho}$ need not be finite everywhere). We form the function $u^{\rho}(r,t)$ by replacing $R(\sigma)$ by $\rho(\sigma)$ in (14) and (15). That is,

$$(18) \quad u^{\rho}(r,t) = \frac{1}{2}\int_0^t \left(\frac{\rho(\sigma)}{t-\sigma} z - \frac{(1+a^{-1})\rho(\sigma)\dot{\rho}(\sigma)+1}{\sqrt{t-\sigma}} \right) \eta(z)\,d\sigma \quad .$$

This function is defined here for both $r > \rho(t)$ and $r < \rho(t)$, and for $t > 0$. Observing that

$$\frac{1}{2}\left[\frac{\rho(\sigma)}{t-\sigma} z - \frac{1+\rho(\sigma)\dot{\rho}(\sigma)}{\sqrt{t-\sigma}}\right]\eta(z) = r\,\eta(z) z_{\sigma} + (\sqrt{t-\sigma}\,\eta(z))_{\sigma} \quad ,$$

$$0 < \sigma < t \quad ,$$

we obtain

$$(19) \quad u^{\rho}(r,t) = r\int_{z(r,0)}^{z(r,t)} \eta(\tau)\,d\tau - \sqrt{t}\,\eta(z(r,0))$$

$$- \frac{1}{2a}\int_0^t \frac{\rho(\sigma)\dot{\rho}(\sigma)}{\sqrt{t-\sigma}}\eta(z)\,d\sigma \quad .$$

Upon differentiating (19) we find

$$(20) \quad u^{\rho}_{r}(r,t) = \int_{z(r,0)}^{z(r,t)} \eta(\tau)\,d\tau - \frac{1}{2\sqrt{t}}\eta(z(r,0))$$

$$+ \frac{1}{2a}\int_0^t \frac{\rho(\sigma)\dot{\rho}(\sigma)}{t-\sigma} z\,\eta(z)\,d\sigma \quad .$$

Since

$$(21) \qquad z(r,t) = \begin{matrix} +\infty & \text{if} \quad r > \rho(t) \\ -\infty & \text{if} \quad r < \rho(t) \end{matrix}$$

u^{ρ}, u^{ρ}_{r} are certainly discontinuous across the line $r = \rho(t)$. We denote the limits of u^{ρ}, u^{ρ}_{r} as $r \longrightarrow \rho(t)$ from the right (left) by $u^{\rho}(\rho^{+},t)$, $u^{\rho}_{r}(\rho^{+},t)$ ($u^{\rho}(\rho^{-},t)$, $u^{\rho}(\rho^{-},t)$).

Theorem 1:

a) u^ρ is a solution of the hear equation for $r < \rho(t)$ and $r > \rho(t)$, and $t > 0$.

b) Expressions (19) and (20) have a meaning for $r = \rho(t)$, which we denote by $u^\rho(\rho,t)$, $u^\rho_r(\rho,t)$.

c) $u^\rho(\rho^\pm,t) = u^\rho(\rho,t) \pm \frac{1}{2}\rho(t)$

$u^\rho_r(\rho^\pm,t) = u^\rho_r(\rho,t) \pm \frac{1}{2}(\alpha^{-1}\rho(t)\dot{\rho}(t) + 1)$.

d) $u^\rho \longrightarrow 0$ as $t \longrightarrow 0$ $(r \neq 0)$, and also as $|r| \longrightarrow \infty$.

We observe that part c) concerning u^ρ follows easily if we assume the continuity of the last term in equation (19). Indeed,

$$u^\rho(\rho^\pm,t) = \rho \int_{z(\rho,0)}^{\pm\infty} \eta(\tau)d\tau + A$$

$$u^\rho(\rho,t) = \rho \int_{z(\rho,0)}^{0} \eta(\tau)d\tau + A \text{ , since } z(r,t) = 0, \text{ for } r = \rho(t)$$

and

$$u^\rho(\rho^\pm,t) - u^\rho(\rho,t) = \rho \int_{0}^{\pm\infty} \eta(\tau)d\tau = \pm \tfrac{1}{2}\rho(t) \quad .$$

A stands for the last two terms in (19) evaluated at $r = \rho(t)$.

We now determine the solution of the equation (22)

(22) $$u^\rho_r(\rho^-,t) = 0 \quad , \rho(0) = 1$$

which solution we denote by $R(t)$.

Theorem 2: Equation (22) has a unique solution $R(t)$ for $0 \leq t \leq t_0$, and this solution is smooth. Furthermore,

$$\dot{R} < 0 \ , \quad 0 < t < t_0 \ , \quad R(t_0) = 0 \ .$$

This solution is constructed in section 3.

Theorem 3:

$$u^R(r,t) \equiv 0 \quad \text{for} \quad r < R(t) \ , \quad t > 0 \ .$$

We now can prove the main

Theorem: $u(r,t) = u^R(r,t)$ for $r > R(t)$ satisfies conditions (5) to (9).

Proof: By Theorem 2, $\dot{R} < 0$, hence Theorem 1 applies to u^ρ with $\rho = R$, and therefore (5) and (8) are satisfied. By Theorem 3, $u^R(r,t) \equiv 0$ for $r < R(t)$, hence in particular, $u^R_r(R^-,t) = 0$. We now have from Theorem 1,

$$u^R(R^+,t) = u^R(R^-,t) + R(t) = R(t)$$

$$u^R_r(R^+,t) = u^R_r(R^-,t) + (\alpha^{-1}\dot{R}(t)R(t) + 1) = \alpha^{-1}R(t)\dot{R}(t) + 1 \quad ,$$

hence conditions (6) and (7) are satisfied. This completes the proof. Obviously, $u^R(r,t)$ is identical with (14) for $r > R(t)$.

To construct the solution for $t > t_0$ as well, we consider the function

$$(23) \quad v(r,t) = r \int_{z(r,0)}^{-\infty} \eta(\tau)d\tau - \sqrt{t}\ \eta(z(r,0)) - \frac{1}{2\alpha} \int_0^{\min\,(t,t_0)} \frac{R(\sigma)\dot{R}(\sigma)}{\sqrt{t-\sigma}} \eta(z)d\sigma \quad ,$$

where $-\infty < r < \infty$, $t > 0$. We show that

$$(24) \quad u = r + v(r,t) - v(-r,t) \ , \quad r > \max\,(0,R(t))$$

satisfies all the conditions (5) through (11). It follows from Theorem 1 that v is a continuous function of r and t for $0 < t \leq t_o$. It can be shown that v is an analytic function of r for $t > t_o$ and a continuous function of r and t for $t \geq t_o$, see [2], ch. XXIX. Clearly, for $t \leq t_o$,

$$v(r,t) = u^R(r,t) \equiv 0 \quad , \qquad \text{for} \quad r < R(t)$$

$$v(r,t) = u^R(r,t) - r = u - r \quad , \text{for} \quad r > R(t) \quad .$$

Hence

$$v(-r,t) \equiv 0 \quad , \qquad \text{for} \quad r > -R(t) \quad ,$$

and therefore for $r > R(t)$ since, $R(t) \geq 0$. Consequently,

$$u \equiv u^R(r,t) \quad , \qquad \text{for} \quad r > R(t) \ , \ t \leq t_o \ ,$$

as required. For $t > t_o$ we use the analyticity property of u. Since u is an odd function of r, $\frac{u}{r}$ is even and analytic. Hence $\frac{u}{r}$ exists as $r \longrightarrow 0$ and $(\frac{u}{r})_r = \dfrac{ru_r - u}{r^2} = 0$ as $r \longrightarrow 0$. This completes the proof.

Explicitly, the final formula for c(r,t) is, after some transformations,

$$(25) \quad c(r,t) = 1 + \left[\int_{z(r,0)}^{-\infty} \left(1 - \frac{2\sqrt{t}}{r}\tau\right)\eta(\tau)d\tau - \frac{1}{2\alpha r}\int_0^{\min\,(t,t_o)} \frac{R(\sigma)\,\dot{R}(\sigma)}{\sqrt{t-\sigma}}\,\eta(z)d\sigma \right]$$

$$+ \left[\int_{z(-r,0)}^{-\infty} \left(1 + \frac{2\sqrt{t}}{r}\tau\right)\eta(\tau)d\tau + \frac{1}{2\alpha r}\int_0^{\min\,(t,t_o)} \frac{R(\sigma)\,\dot{R}(\sigma)}{\sqrt{t-\sigma}}\,\eta(z)d\sigma \right] .$$

Here $t > 0$, $r > \max\,(0,R(t))$, and R(t) is determined by equation (22). The second bracket is identically zero for $t \leq t_o$.

3. Determination of the evaporation curve

The evaporation curve is determined by equation (22) of the preceding section. In view of Theorem 1, this equation is

$$(26) \qquad u_r^\rho(\rho,t) = \frac{1}{2}\left[\alpha^{-1}\rho(t)\dot{\rho}(t) + 1\right] \quad , \quad \rho(0) = 1$$

or explicitly

$$(27) \quad \frac{1}{2}\left[\alpha^{-1}\rho(t)\dot{\rho}(t) + 1\right] = \int_{z(\rho,0)}^{0} \eta(\tau)\,d\tau - \frac{1}{2\sqrt{t}}\,\eta(z(\rho,0)) + \frac{1}{2\alpha}\int_0^t \frac{\rho(\sigma)\,\dot{\rho}(\sigma)}{t-\sigma}\, z\,\eta(z)\,d\sigma \quad .$$

After some transformations this can be reduced to a simpler form

$$(28) \qquad \begin{cases} \rho\dot{\rho} = G(\rho) \\ \rho(0) = 1 \end{cases}$$

where

$$(29) \quad G(\rho) = -2\alpha \int_{-\infty}^{z(\rho,0)} \left(1 - \frac{\tau}{\sqrt{t}}\right)\eta(\tau)\,d\tau + \int_0^t \frac{\rho(\sigma)\,\dot{\rho}(\sigma)}{t-\sigma}\, z(\rho,\sigma)\,\eta(z(\rho,\sigma))\,d\sigma \quad .$$

This equation is best handled by the method of iterations. We first define an auxiliary function $w(t,a,k)$ by

$$(30) \qquad w^2(t,a,k) = k^2 - 2\alpha\left[(t-a) + \frac{2}{\sqrt{\pi}}(\sqrt{t} - \sqrt{a}\,)\right]$$

where

$$(31) \quad a \geq 0,\ 0 < k \leq 1,\ 0 \leq t \leq t(a,k) = \left(-\frac{1}{\sqrt{\pi}} + \sqrt{\frac{w^2(0)}{2\alpha} + \frac{1}{\pi}}\right)^2 .$$

w(t,a,k) passes through the point with coordinates (a,k), and vanishes for t = t(a,k). Furthermore,

$$(32) \qquad ww' = -\alpha\left(1 + \frac{1}{\sqrt{\pi t}}\right) = G(1) \; , \quad t > 0 \; .$$

We now form the sequence of functions $\rho_n(t)$ and the sequence of numbers a_n, k_n as follows:

$$(33) \qquad \rho_0(t) = w(x,0,1) \; , \quad 0 \leq t \leq t(0,1) = a_0$$

$$(33^*) \qquad \rho_{2n+1}(t) = \sqrt{2\left(1 + \int_0^t G(\rho_{2n})dt\right)} \; , \quad 0 \leq t \leq a_{2n+1} \; ,$$

$$k_{2n+1} = \rho_{2n+1}(a_{2n+1}) \; ,$$

where a_{2n+1} is the first zero of $\dot{\rho}_{2n+1}(t)$,

$$(33^{**}) \quad \begin{cases} \rho_{2n+2}(t) = \sqrt{2\left(1 + \int_0^t G(\rho_{2n+1})dt\right.} \; , & 0 \leq t \leq a_{2n+1} \\ \qquad = w(t,a_{2n+1},k_{2n+2}), & a_{2n+1} \leq t \leq t(a_{2n+1},k_{2n+2}) = a_{2n+2} \end{cases}$$

where $k_{2n+2} = \rho_{2n+2}(a_{2n+1})$.

Theorem: If $\alpha \leq \dfrac{1}{1 + \sqrt{8/\pi}}$,

a) The sequence a_n converges to a finite number t_0

b) The sequence k_n converges to zero.

c) For any $\bar{t} < t_0$ there is an N such that for $n \geq N$ each ρ_n is defined for all $0 \leq t \leq \bar{t}$ and the sequence ρ_n converges to a function R(t) which is continuously differentiable (except at t = 0). R(t) is the unique solution of (28). Furthermore,

$$\lim_{t \to t_0} R(t) = 0 \; , \quad \dot{R}(t) < 0 \; .$$

d) For any n,

$$\rho_{2n} \leq \rho_{2n+2} \leq R \leq \rho_{2n+3} \leq \rho_{2n+1}$$

$$\dot{\rho}_{2n} < \dot{\rho}_{2n+2} < \dot{R} < \rho_{2n+3} < \dot{\rho}_{2n+1}$$

whenever any of two compared functions are simultaneously defined.

In view of this theorem, the evaporation curve is constructed over the full period of evaporation by means of a single sequence of functions. Furthermore, in view of part d), any two consecutive iterates yield lower and upper bounds for the solution and its derivative.

4. Estimates for R(t)

Using part d) of the theorem of the preceding section, we have, for any n,

$$(34) \qquad \dot{\rho}_{2n} < \dot{R} < \dot{\rho}_{2n+1} , \quad \rho_{2n} \leq R \leq \rho_{2n+1} .$$

Here, the first part of the two inequalities holds for $0 \leq t \leq a_{2n}$, whereas the second part holds for $0 \leq t \leq a_{2n+1}$. In particular, if α is small, one gets simple estimates which are sufficiently sharp for all practical purposes by taking $n = 0$ in (34). Thus we get, with $w = w(t,0,1)$, (see equation (30))

$$(35) \qquad R \geq w , \quad \dot{R} \geq \dot{w} , \quad 0 \leq t \leq a_o$$

$$(35^*) \quad R < \sqrt{2(1 + \int_0^t G(w)dt)} , \quad \dot{R} < \frac{G(w)}{\sqrt{2(1 + \int_0^t G(w)dt)}} ,$$

$$0 \leq t \leq a_1 < a_o .$$

The numbers a_{2n}, and in particular a_o yield lower bounds for t_o. None of the above estimates yields an upper bound for t_o. Such a bound, together with sharp upper bound for $R(t)$ (though not for $\dot{R}(t)$) is obtained in a strikingly simple way directly from the formulation of the problem. As is well known, see e.g. [2], ch. XXIX, for any regular solution $u(r,t)$ of the heat equation in a closed domain D,

$$(36) \qquad \oint (u_r dt + u dr) = 0 \quad .$$

We apply this identity to a strip bounded by the derivatives $t = t'$, $t = 0$, and the evaporation curve, obtaining

$$(27) \qquad \int_{R(t')}^{\infty} u(r,t')dr = -\int_0^{t'} (u_r(R,t)+u(R,t)\dot{R}(t))dt + \int_1^{\infty} u(r,0)dr$$

$$= -t' - \frac{\alpha+1}{2\alpha}(R^2(t')-1), \quad 0 \leq t' \leq t_o \quad .$$

Now, it can be shown by using the minimum-maximum principle for solutions of the heat equation, see [3], that $n \geq 0$ for all t and r *. Hence we get

$$(38) \qquad R^2 < 1 - \frac{2\alpha}{\alpha+1} t, \quad 0 \leq t \leq t_o \quad ,$$

and since $R \geq 0$,

$$(39) \qquad t_o < \frac{\alpha+1}{2\alpha} \quad .$$

These bounds have been discussed in the introduction.

---

[*] This is obvious on physical grounds, since n represents a density concentration.

5. Remarks on the method of solution

The main advantage of the method used here to solve the equation for R(t) is that it yields in a single sequence a solution over the full range of the evaporation time, and thus permits an estimate of the evaporation time. Moreover, this sequence is alternating so that any two successive iterates yield upper and lower bounds for the solution.

The final expression for the concentration, equation (25) could be derived by evaluating first the Laplace transform of the concentration and inverting. Since to determine the Laplace transform one has to solve a second order ordinary differential equation with constant coefficients, and since the inversion can be carried out with the help of existing tables, such a procedure would seem at first simpler, and, especially more direct than the one used here. Actually one would thus obtain only a formal expression which it is difficult to handle. We would have at our disposal the formula (25) from which R(t) would have to be determined in some way. The matter would be complicated by the appearance of the second bracket in (25) which as we know now is identically zero during the evaporation time, but which would persist in an equation determining R(t) if we use equation (25) as the starting point. This may be why no adequate treatment of this apparently simple and fundamental problem has appeared until now. The method used here avoids the difficulty and leads to equation (25) as an end-product and not as a starting point.

We note, without further elucidation, that the method illustrated here can and has been successfully used in handling various problems of similar structure, such as problems of change of phase with unknown (free) boundaries. See [4].

References

[1] Keller, J. B., Kolodner, I., and Ritger, P., Decay of Drops by Evaporation and Growth by Condensation, Institute for Mathematics and Mechanics, New York University, July 1952.

[2] Goursat, E., Cours d'Analyse Mathematique, Vol. III, Ch. XXIX. (Fifth Edition, Gauthier-Villars, Paris, 1942.)

[3] Nirenberg, L., A strong maximum principle for parabolic equations. Communications on Pure and Applied Mathematics, Vol. VI, p. 167-177, (1953).

[4] Kolodner, I., A method for solving the heat equation with moving boundary condition. Abstract, to appear in the Bulletin of the American Mathematical Society.

Date Due

7/25			
OCT 28 '58			
3/20/61			
MAY 9 '61			
8/15/61			
11/20/62			
12/3/62			
APR 5 '62			
1.14 64			
JUN 2 '64			
[illegible] 10 [illegible]			
MAR 16 '68	ILL		
Bell Tel Labs, Inc			
6 June	Brookhaven		
SEP [illegible] '68			
	PRINTED	IN U. S. A.	

NYU c.1
IMM
199 Kolodner

NYU c.1
IMM
199 Kolodner
AUTHOR
Decay of drops by
TITLE
evaporation.

c.1

~~NEW YORK UNIVERSITY~~
~~INSTITUTE OF MATHEMATICAL SCIENCES~~
LIBRARY
~~25 Waverly Place, New York 3, N.Y.~~

New York UNIVERSITY
COURANT INSTITUTE - LIBRARY
251 Mercer St. New York, N.Y. 10012

KOLODNER

NYU IMM

199